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Positivity of Narayana polynomials and Eulerian polynomials

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A B S T R A C T

Gamma-positive polynomials frequently appear in finite geometries, algebraic combinatorics and number theory. Sagan and Tirrell (2020) [34] stumbled upon some unimodal sequences, which turn out to be alternating gamma-positive instead of gamma-positive. Motivated by this work, we first show that one can derive alternatingly γ-positive polynomials from γ-positive polynomials. We then prove the alternating γ-positivity and Hurwitz stability of several polynomials associated with the Narayana polynomials of types A and B. In particular, by introducing the definition of colored $2 \times n$ Young diagrams, we provide combinatorial interpretations for three identities related to the Narayana numbers of type B. Finally, we present several identities involving the Eulerian polynomials of types A and B.

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Contents

1. Introduction

In this paper we study the (alternating) γ-positivity of several polynomials associated with the Narayana and Eulerian polynomials. In particular, we provide combinatorial interpretations for three identities related to the Narayana numbers of type B. Moreover, we show the alternating γ-positivity and Hurwitz stability of the polynomials $N(B_n,x^2)+(n+1)xN(A_{n-1},x^2)$, where $N(A_{n-1},x)$ and $N(B_n,x)$ are the Narayana polynomials of types A and B, respectively.

In subsection 1.1, we collect the definitions, notation and preliminary results. In subsection 1.2, we outline the motivations and the organization of this paper.

1.1. Notation and preliminaries

Let $f(x)=\sum_{i=0}^n f_ix^i\in\mathbb{R}[x]$. If $f_0\leqslant f_1\leqslant\cdots\leqslant f_k\geqslant f_{k+1}\geqslant\cdots\geqslant f_n$ for some k, then $f(x)$ is said to be *unimodal*, where the index k is called the *mode* of $f(x)$. If $f(x)$ is symmetric with the centre of symmetry $\lfloor n/2\rfloor$, i.e., $f_i=f_{n-i}$ for all $0\leqslant i\leqslant n$, then it can be expanded as

$$f(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\gamma_kx^k(1+x)^{n-2k}.$$

Following Gal [17], the polynomial $f(x)$ is *γ-positive* if $\gamma_k\geqslant 0$ for all $0\leqslant k\leqslant\lfloor n/2\rfloor$, and the sequence $\{\gamma_k\}_{k=0}^{\lfloor n/2\rfloor}$ is called the *γ-vector* of $f(x)$. Clearly, γ-positivity implies symmetry and unimodality. The reader is referred to [1,7,25,27,29,35,38] for some recent progress on this subject.

The polynomial $f(x)$ is said to be *alternatingly γ-positive* if the γ-vector of $f(x)$ alternates in sign. For example, $(1+x^2)^n$ is alternatingly γ-positive, since

$$(1+x^2)^n=[(1+x)^2-2x]^n=\sum_{k=0}^n\binom{n}{k}2^k(-x)^k(1+x)^{2n-2k},$$

where the coefficients $\binom{n}{k}2^k$ count k-simplices in the n-cube (see [36, A013609]). There has been considerable recent interest in the study of alternatingly γ-positive polynomials, see [6,21,24,34] for instance. In particular, Lin et al. [21] studied the alternating γ-positivity of alternating Eulerian polynomials. Let us now recall two well known formulas:

$$p^n + q^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(pq)^k(p+q)^{n-2k}, \tag{1}$$

$$\sum_{i=0}^{n} p^i q^{n-i} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}(pq)^k(p+q)^{n-2k}. \tag{2}$$

There are several applications of the above two formulas, see [6, Section 3], [13, p. 156] and [19, p. 1068]. Based on the structures of matchings on path and cycle graphs, Brittenham et al. [6] provided combinatorial interpretations for the alternating γ-expansions of $1+q^n$ and $\sum_{i=0}^{n} q^i$.

As usual, we use $\mathfrak{S}_n$ to denote the symmetric group of all permutations of $[n] = \{1,2,\ldots,n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. In this paper, we always assume that $\pi(0) = \pi(n+1) = \infty$ (except where explicitly stated). If $i \in [n]$, then $\pi(i)$ is called

- a *descent* if $\pi(i) > \pi(i+1)$;
- an *ascent* if $\pi(i) < \pi(i+1)$;
- a *peak* if $\pi(i-1) < \pi(i) > \pi(i+1)$;
- a *valley* if $\pi(i-1) > \pi(i) < \pi(i+1)$;
- a *double descent* if $\pi(i-1) > \pi(i) > \pi(i+1)$;
- a *double ascent* if $\pi(i-1) < \pi(i) < \pi(i+1)$.

Let $\operatorname{des}(\pi)$ (resp. $\operatorname{asc}(\pi)$, $\operatorname{pk}(\pi)$, $\operatorname{val}(\pi)$, $\operatorname{ddes}(\pi)$, $\operatorname{dasc}(\pi)$) denote the number of descents (resp. ascents, peaks, valleys, double descents, double ascents) of π. Moreover, for $i \in [n-1]$, we say that $\pi(i)$ is a *left peak* if $\pi(i-1) < \pi(i) > \pi(i+1)$, where we set $\pi(0) = 0$. Let $\operatorname{lpk}(\pi)$ be the number of left peaks of π.

The *type A Eulerian polynomials* are defined by

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)}.$$

Let $\gamma_{n,k} = \#\{\pi \in \mathfrak{S}_n : \operatorname{pk}(\pi) = k, \operatorname{ddes}(\pi) = 0\}$. Foata-Schützenberger [14] found that

$$A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} x^k (1+x)^{n-1-2k} \text{ for } n \geqslant 1, \tag{3}$$

which has been extensively studied in the past decades, see [7,11,38] and references therein. For example, using the theory of enriched P-partitions, Stembridge [37, Remark 4.8] found that

$$A_n(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 4^k P(n,k) x^k (1+x)^{n-1-2k}, \tag{4}$$

where $P(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k peaks.

Let $\pm[n] = [n] \cup \{-1,-2,\ldots,-n\}$, and let B_n be the hyperoctahedral group of rank n. Elements of B_n are signed permutations of $\pm[n]$ with the property that $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$. The *type B Eulerian polynomials* are defined by

$$B_n(x) = \sum_{\sigma \in B_n} x^{\mathrm{des}_B(\sigma)},$$

where $\mathrm{des}_B(\sigma) = \#\{i \in \{0,1,2,\ldots,n-1\} : \ \sigma(i) > \sigma(i+1)\}$ and $\sigma(0) = 0$ (see [5] for details). Using the theory of enriched P-partitions, Petersen [28, Proposition 4.15] obtained that

$$B_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} 4^i \widehat{P}(n,i) x^i (1+x)^{n-2i}, \tag{5}$$

where $\widehat{P}(n,i)$ is the number of permutations in $\mathfrak{S}_n$ with i left peaks.

For convenience, we collect the following recursions (see [26, p. 1] for instance):

$$\begin{aligned} A_n(x) &= (nx+1-x)A_{n-1}(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_{n-1}(x),\ A_0(x) = 1; \\ B_n(x) &= (2nx+1-x)B_{n-1}(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_{n-1}(x),\ B_0(x) = 1. \end{aligned} \tag{6}$$

Let Δ be a simplicial complex of dimension $n-1$. The *f-vector* of Δ is the sequence of integers $(f_{-1}, f_0, f_1, \ldots, f_{n-1})$, where f_i is the number of faces with $i+1$ vertices in Δ. For example, $f_{-1} = 1$, corresponding to the empty face. The *f-polynomial* and *h-polynomial* of Δ are respectively defined as $f(x) = \sum_{i=0}^{n} f_{i-1}x^i$, and

$$h(x) = (1-x)^n f\left(\frac{x}{1-x}\right) = \sum_{i=0}^{n} f_{i-1}x^i(1-x)^{n-i} = \sum_{i=0}^{n} h_i x^i.$$

The sequence $(h_0, h_1, \ldots, h_n)$ is called the *h-vector* of Δ. It is well known that the h-polynomial of a simple polytope is positive and symmetric [30]. In [16], Fomin-Zelevinsky defined the (generalized) Narayana numbers $N_k(\Phi)$ for an arbitrary root system Φ as the entries of the h-vector of the simplicial complex dual to the corresponding generalized associahedron. Let $N(\Phi, x) = \sum_{k=0}^{n} N_k(\Phi)x^k$. For the classical Weyl groups, the generating polynomials for the Narayana numbers are given as follows (see [4, Section 7]):

$$N(A_n,x)=\sum_{k=0}^{n}\frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}x^k,$$
$$N(B_n,x)=\sum_{k=0}^{n}\binom{n}{k}^2x^k,$$
$$N(D_n,x)=N(B_n,x)-nxN(A_{n-2},x),$$

where $A_n=\mathfrak{S}_{n+1}$ and D_n is the group of even-signed permutations in B_n. Narayana polynomials possess many of the same or similar properties as Eulerian polynomials (see [22,25,29]), including real-rootedness, γ-positivity and combinatorial interpretations.

1.2. The motivations and the organization of the paper

The *Lucas polynomials* $\{n\}:=\{n\}_{s,t}$ are defined by $\{n\}=s\{n-1\}+t\{n-2\}$ with the initial conditions $\{0\}=0,\ \{1\}=1$. When $s=1+q,t=-q$, one has $\{n\}=1+q+q^2+\cdots+q^{n-1}$. Sagan-Tirrell [34] introduced a sequence of polynomials $P_n(s,t)$ by using the factorization of $\{n\}$: $\{n\}=\Pi_{d|n}P_d(s,t)$. The polynomials $P_n(s,t)$ are called *Lucas atoms*. They found that the coefficients of $P_n(s,t)$ are just the absolute values of the γ-coefficients of the cyclotomic polynomials $\Phi_n(q)=\Pi_\zeta(q-\zeta)$, where the product is over all primitive nth roots of unity. Motivated by the work of Sagan-Tirrell [34], in Section 2, we will present Theorem 2.

Set $D=\frac{\mathrm{d}}{\mathrm{d}x}$. It is well known (see [20]) that

$$(xD)^n\frac{1}{1-x}=\sum_{k=0}^{\infty}k^nx^k=\frac{xA_n(x)}{(1-x)^{n+1}}.$$

When $n\geqslant 1$, using (6), we find that

$$(xD)^n\frac{1}{1-x^2}=\frac{2^nx^2A_n(x^2)}{(1-x^2)^{n+1}},\ \ (xD)^n\frac{x}{1-x^2}=\frac{xB_n(x^2)}{(1-x^2)^{n+1}},\tag{7}$$

which can be proved by induction. Therefore, using the fact that

$$(xD)^n\frac{1}{1-x}=(xD)^n\frac{1}{1-x^2}+(xD)^n\frac{x}{1-x^2},$$

we get

$$(1+x)^{n+1}A_n(x)=B_n(x^2)+2^nxA_n(x^2).\tag{8}$$

The idea underlying this proof is very simple, but it offers a new insight. It is natural to explore similar expressions of Narayana polynomials of types A and B. Perhaps the most straightforward proof of (8) is by generating functions, see [23, Theorem 3] for instance.

In Sections 3 and 4, we shall give several applications of Theorem 2. Moreover, inspired by (7) and (8), we will show the alternating γ-positivity and Hurwitz stability of several polynomials related to the Narayana polynomials of types A and B. To sum up, the main results of this paper are Theorems 2, 4, 10, 12 and 17.

2. Relationship between gamma-positivity and alternating gamma-positivity

Let $f(x)=\sum_{i=0}^{n} f_i x^i$. We define the operator $\mathcal{A}_m:\mathbb{R}[x]\to\mathbb{R}[x]$ by $\mathcal{A}_m(f(x))=f(x^m)$. The operator $\mathcal{A}_m$ frequently appears in the study of field theory and number theory (see [18,26,33]). For example, there are two reduction formulas of cyclotomic polynomials (see [34, Theorem 5.1]): $\Phi_p(q)=\sum_{i=0}^{p-1} q^i,\ \Phi_{pn}(q)=\frac{\Phi_n(q^p)}{\Phi_n(q)}$, where $n\in\mathbb{N}$ and p is a prime not dividing n.

Lemma 1. *The product of two alternatingly γ-positive polynomials is alternatingly γ-positive.*

Proof. Let $f(x)$ and $g(x)$ be two alternatingly γ-positive polynomials. Suppose that

$$f(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\gamma_k(-x)^k(1+x)^{n-2k},\ g(x)=\sum_{\ell=0}^{\lfloor m/2\rfloor}\eta_\ell(-x)^\ell(1+x)^{m-2\ell},$$

where $\gamma_k\geqslant 0$ for $0\leqslant k\leqslant\lfloor n/2\rfloor$ and $\eta_\ell\geqslant 0$ for $0\leqslant \ell\leqslant\lfloor m/2\rfloor$. Then

$$f(x)g(x)=\sum_{i=0}^{\lfloor (n+m)/2\rfloor}\sum_{k=0}^{i}\gamma_k\eta_{i-k}(-x)^i(1+x)^{n+m-2i},$$

as desired. This completes the proof. □

We can now conclude the first main result of this paper.

Theorem 2. *Let $f(x)=\sum_{i=0}^{n} f_i x^i\in\mathbb{R}[x]$. Assume that $f(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\gamma_i x^i(1+x)^{n-2i}$.*

(i) If $f(x)$ is γ-positive, then $\mathcal{A}_{2m}(f(x))=f(x^{2m})$ is alternatingly γ-positive, where $m\in\mathbb{N}$.
(ii) Set $\eta_k=\sum_{i=0}^{\lfloor k/2\rfloor}\binom{n-2i}{k-2i}2^{k-2i}\gamma_i$. Then we have

$$f(x^2)=\sum_{k=0}^{n}\eta_k(-x)^k(1+x)^{2n-2k}. \tag{9}$$

Since $x^2=(-x)^2$, consequently, $f(x^2)=\sum_{k=0}^{n}\eta_k x^k(1-x)^{2n-2k}$. Moreover, the following two identities are equivalent:

$$\sum_{i=0}^{n} f_i x^{2i} = \sum_{k=0}^{n} \eta_k (-x)^k (1+x)^{2n-2k}, \quad \sum_{i=0}^{n} f_i x^{2i} (1+x)^{2n-2i} = \sum_{k=0}^{n} \eta_k x^k (1+x)^k. \tag{10}$$

(iii) Setting $\xi_k = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-2i}{k-2i} \gamma_i$*, we have*

$$\sum_{k=0}^{n} \eta_k x^k = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2i} (1+2x)^{n-2i} = \sum_{k=0}^{n} \xi_k x^k (1+x)^{n-k}. \tag{11}$$

(iv) The modified γ*-coefficient polynomial of* $f(x)$ *has two equivalent expansions:*

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2i} = \sum_{k=0}^{n} \xi_k (-x)^k (1+x)^{n-k} = \sum_{k=0}^{n} \xi_k x^k (1-x)^{n-k}. \tag{12}$$

Proof. (i) Using (1), we get $1 + x^{2m} = \sum_{j=0}^{m} a_{2m,j} (-x)^j (1+x)^{2m-2j}$, where $a_{m,j} = \frac{m}{m-j} \binom{m-j}{j}$. So we have

$$f(x^{2m}) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2mi} [1 + x^{2m}]^{n-2i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2mi} \left\{ \sum_{j=0}^{m} a_{2m,j} (-x)^j (1+x)^{2m-2j} \right\}^{n-2i}.$$

Using Lemma 1, we obtain

$$\begin{aligned} f(x^{2m}) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2mi} \sum_{\ell=0}^{m(n-2i)} b_{m,\ell} (-x)^{\ell} (1+x)^{2mn-4mi-2\ell} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{\ell=0}^{m(n-2i)} \gamma_i b_{m,\ell} (-x)^{2mi+\ell} (1+x)^{2mn-2(2mi+\ell)} \\ &= \sum_{k=0}^{mn} \left(\sum_{2mi+\ell=k} \gamma_i b_{m,\ell} \right) (-x)^k (1+x)^{2mn-2k}, \end{aligned}$$

where

$$b_{m,\ell} = \sum_{(i_0,i_1,i_2,\ldots,i_m)} \frac{(n-2i)!}{i_0! i_1! i_2! \cdots i_m!} a_{2m,0}^{i_0} a_{2m,1}^{i_1} a_{2m,2}^{i_2} \cdots a_{2m,m}^{i_m},$$

and the summation is over all sequences of nonnegative integers $(i_0, i_1, i_2, \ldots, i_m)$ such that $\sum_{j=1}^{m} j i_j = \ell$ and $\sum_{j=0}^{m} i_j = n - 2i$. Thus $f(x^{2m})$ is alternatingly γ-positive.

(*ii*) Note that

$$\begin{aligned} f(x^2) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2i}[(1+x)^2 - 2x]^{n-2i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{\ell=0}^{n-2i} 2^\ell \binom{n-2i}{\ell} \gamma_i (-x)^{2i+\ell}(1+x)^{2n-2(2i+\ell)} \\ &= \sum_{k=0}^{n} \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-2i}{k-2i} 2^{k-2i} \gamma_i (-x)^k (1+x)^{2n-2k}, \end{aligned}$$

and this proves (9). Clearly, one has $\sum_{i=0}^{n} f_i x^{2i}(1+x)^{2n-2i} = (1+x)^{2n} \sum_{i=0}^{n} f_i \left(\frac{x}{1+x}\right)^{2i}$. Recall that $f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$. Substituting x by $\frac{x^2}{(1+x)^2}$, we deduce that

$$\begin{aligned} (1+x)^{2n} \sum_{i=0}^{n} f_i \left(\frac{x}{1+x}\right)^{2i} &= (1+x)^{2n} \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i \left(\frac{x}{1+x}\right)^{2i} \left(1 + \frac{x^2}{(1+x)^2}\right)^{n-2i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i (x(1+x))^{2i} (1+2x(1+x))^{n-2i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{\ell=0}^{n-2i} \binom{n-2i}{\ell} 2^\ell \gamma_i (x(1+x))^{2i+\ell} \\ &= \sum_{k=0}^{n} \left\{ \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-2i}{k-2i} 2^{k-2i} \gamma_i \right\} x^k (1+x)^k. \end{aligned}$$

Therefore, we obtain $\sum_{i=0}^{n} f_i x^{2i}(1+x)^{2n-2i} = \sum_{k=0}^{n} \eta_k x^k (1+x)^k$. This proves (10).

(*iii*) On the one hand, we have

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2i}(1+2x)^{n-2i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{\ell=0}^{n-2i} \binom{n-2i}{\ell} 2^\ell \gamma_i x^{2i+\ell} = \sum_{k=0}^{n} \eta_k x^k.$$

On the other hand, since $1 + 2x = 1 + x + x$, we get

$$\begin{aligned} \sum_{k=0}^{n} \eta_k x^k &= \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^{2i}(1+x+x)^{n-2i} \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{\ell=0}^{n-2i} \binom{n-2i}{\ell} \gamma_i x^{2i+\ell}(1+x)^{n-2i-\ell} \\ &= \sum_{k=0}^{n} \xi_k x^k (1+x)^{n-k}. \end{aligned}$$

This proves (11).

(*iv*) Making the substitution $\frac{x}{1+2x} = y$, it follows from (11) that

$$\begin{aligned}\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i y^{2i} &= (1-2y)^n \sum_{k=0}^{n} \eta_k \left(\frac{y}{1-2y}\right)^k \\ &= (1-2y)^n \sum_{k=0}^{n} \xi_k \left(\frac{y}{1-2y}\right)^k \left(\frac{1-y}{1-2y}\right)^{n-k} \\ &= \sum_{k=0}^{n} \xi_k y^k (1-y)^{n-k}.\end{aligned}$$

Since $y^2 = (-y)^2$, we get $\sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i y^{2i} = \sum_{k=0}^{n} \xi_k (-y)^k (1+y)^{n-k}$. This proves (12). □

Combining (4), (5) and (9), we get the following result.

Proposition 3. *For any $n \geqslant 1$, we have*

$$A_n(x^2) = \sum_{k=0}^{n-1} \frac{1}{2^{n-1-k}} \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-1-2i}{k-2i} P(n,i)(-x)^k (1+x)^{2n-2-2k},$$

$$B_n(x^2) = \sum_{k=0}^{n} 2^k \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-2i}{k-2i} \widehat{P}(n,i)(-x)^k (1+x)^{2n-2k}.$$

If $f(x)$ is γ-positive, then $\mathcal{A}_{2m+1}(f(x)) = f(x^{2m+1})$ may be not alternatingly γ-positive. For example, if $f(x) = 1 + 4x + x^2$, then $f(x) = (1+x)^2 + 2x$ and

$$f(x^3) = (1+x)^6 - 6x(1+x)^4 + 9x^2(1+x)^2 + 2x^3.$$

Thus $f(x)$ is γ-positive, but $f(x^3)$ is not alternatingly γ-positive.

In the next two sections, we shall give several applications of Theorem 2.

3. Narayana polynomials

3.1. Identities involving Narayana polynomials

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the *Catalan numbers*. It is well known that Catalan numbers and the *central binomial coefficients* have the following expressions (see [8,12,31]):

$$C_n = \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k+1} \binom{n}{k}, \quad \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Using generating functions and Lagrange inversion formula, Coker [12] derived that

$$\sum_{k=0}^{n}\frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}x^k=\sum_{k=0}^{\lfloor n/2\rfloor}C_k\binom{n}{2k}x^k(1+x)^{n-2k},$$

$$\sum_{k=0}^{n}\frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k}x^{2k}(1+x)^{2n-2k}=\sum_{k=0}^{n}C_{k+1}\binom{n}{k}x^k(1+x)^k. \tag{13}$$

Chen-Yan-Yang [9] gave combinatorial interpretations of these two identities based on a bijection between Dyck paths and 2-Motzkin paths. In [32, p. 81], Riordan derived that

$$\sum_{k=0}^{n}\binom{n}{k}^2x^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}x^k(1+x)^{n-2k}. \tag{14}$$

Using weighted type B noncrossing partitions, Chen-Wang-Zhao [8] proved that

$$\sum_{k=0}^{n}\binom{n}{k}^2x^{2k}(1+x)^{2n-2k}=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k(1+x)^k. \tag{15}$$

Combining (10), (11), (13) and (15), we get the following result.

Theorem 4. *For $n\geqslant 0$, one has*

$$N(A_n,x^2)=\sum_{k=0}^{n}C_{k+1}\binom{n}{k}(-x)^k(1+x)^{2n-2k}, \tag{16}$$

$$N(B_n,x^2)=\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}(-x)^k(1+x)^{2n-2k}. \tag{17}$$

Thus $N(A_n,x^2)$ and $N(B_n,x^2)$ are both alternatingly γ-positive. Moreover,

$$\sum_{k=0}^{n}C_{k+1}\binom{n}{k}x^k=\sum_{k=0}^{\lfloor n/2\rfloor}C_k\binom{n}{2k}x^{2k}(1+2x)^{n-2k},$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}x^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}x^{2k}(1+2x)^{n-2k}.$$

Corollary 5. *For any $n\geqslant 2$, one has*

$$N(D_n,x^2)=(1+x)^{2n}+\sum_{i=1}^{n}\left(\binom{n}{i}\binom{2i}{i}-nC_{i-1}\binom{n-2}{i-2}\right)(-x)^i(1+x)^{2n-2i},$$

and thus $N(D_n,x^2)$ is alternatingly γ-positive.

Proof. The alternating γ-expansion of $N(D_n, x^2)$ follows from the fact that

$$N(D_n, x^2) = N(B_n, x^2) - nx^2 N(A_{n-2}, x^2).$$

When $i = 1$, $\binom{n}{i}\binom{2i}{i} - nC_{i-1}\binom{n-2}{i-2} = 2n$, and for any $2 \leqslant i \leqslant n$, we have

$$\frac{\binom{n}{i}\binom{2i}{i}}{nC_{i-1}\binom{n-2}{i-2}} = \frac{2(n-1)(2i-1)}{i(i-1)} = 2(n-1)\left(\frac{1}{i} + \frac{1}{i-1}\right) \geqslant 0. \quad \square$$

3.2. The combinatorial proofs of (14)*,* (15)*,* (16)*,* (17)

A *Motzkin path* is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, and never going below the x-axis, with three possible steps $(1, 1)$, $(1, 0)$ and $(1, -1)$. As usual, we use U, D and H to denote an up step $(1, 1)$, a down step $(1, -1)$ and a horizontal step $(1, 0)$, respectively. For any $c \in \mathbb{N}$, a *c-Motzkin path* is a Motzkin path with the horizontal steps can be colored by one of c colors. When $c = 0$, there are no horizontal steps and 0-Motzkin paths reduce to Dyck paths. When $c = 1$, c-Motzkin paths reduce to the ordinary Motzkin paths. When $c = 2$, a horizontal step may be B or R, where B and R stand for a blue step and a red step, respectively. When $c = 3$, a horizontal step may be B, R or G, where G denotes a green step. The *length* of a lattice path is defined to be the number of steps. The weight of a path is defined to be the product of the weights of its steps, and the weight of a set of paths equals the sum of weights of its paths.

Chen-Yan-Yang [9] discovered a fundamental result.

Lemma 6 *([9, Lemma 3.4]). Let* CM_n *be the set of* 2*-Motzkin paths of length* n*. One has*

$$\frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} = \#\{P \in \mathrm{CM}_n : \ \mathrm{UB}\,(P) = k\},$$

where $\mathrm{UB}\,(P)$ *denote the total number of* U *and* B *steps on* P*. Thus* $\#\,\mathrm{CM}_n = C_{n+1}$*.*

Combinatorial proof of the identity (16)**:**
For any path in CM_n, we assign the weight x^2 to each U or B step and the weight 1 to any other step. By Lemma 6 (or refer back to [9, Lemma 3.4]), the left-hand side of (16) equals the weight of CM_n. It should be noted that the steps of U's and D's must be matched on any path of CM_n. We use $S(k)$ to denote any subset of CM_n with k up steps and the up and down steps are all located in given positions, see Fig. 1 for illustrations. Then the weight of $S(k)$ equals $x^{2k}(1 + x^2)^{n-2k}$, since a blue step has the weight x^2 and a red step has the weight 1.

Let TM_n denote the set of 3-Motzkin paths of length n. For any path in TM_n, we assign the weight $(-x)$ to each of the U, D, B and R steps, and the weight $(1 + x)^2$ to each G step. We use $\widehat{S}(k)$ to denote any subset of TM_n with k up steps and the up and down steps are all located in given positions, see Fig. 2 for an example. Since

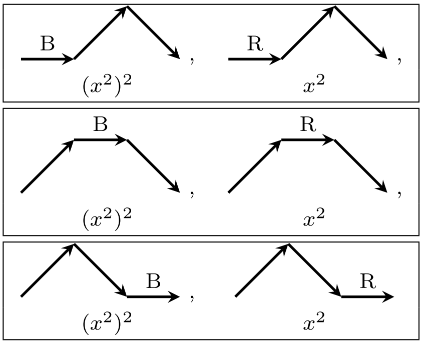

Fig. 1. The subset $S(1)$ in CM_3 with weight $3x^2(1+x^2)$.

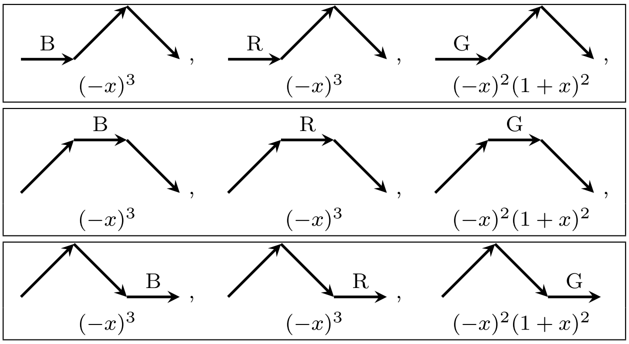

Fig. 2. The subset $\widehat{S}(1)$ in TM_3 with weight $3x^2(1+x^2)$.

$x^2 = (-x)(-x),\ 1+x^2 = (1+x)^2 - x - x$, the weight of $\widehat{S}(k)$ equals

$$x^{2k}\left((1+x)^2 - x - x\right)^{n-2k} = x^{2k}(1+x^2)^{n-2k},$$

which shows that $\widehat{S}(k)$ and $S(k)$ have the same weight. It remains to show that the weight TM_n coincides with the right-hand side of (16). To construct a path of TM_n with $n-k$ G steps, we can insert the G steps into 2-Motzkin paths of CM_k, where the U, D, B and R steps all have the same weight $(-x)$. Note that there are $\binom{n}{n-k} = \binom{n}{k}$ ways to insert the G steps. By Lemma 6, $\#\,\mathrm{CM}_k = C_{k+1}$. So the weight of TM_n equals

$$\sum_{k=0}^{n}\binom{n}{k}\left((1+x)^2\right)^{n-k}C_{k+1}(-x)^k = \sum_{k=0}^{n}C_{k+1}\binom{n}{k}(-x)^k(1+x)^{2n-2k}.$$

This completes the proof. □

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$, we draw a left-justified array with λ_i cells in the i-th row. This array is called the *Young diagram* of λ. The partition that is represented by such a diagram is said to be the shape of the diagram. We will identify a partition λ with its Young diagram. We now introduce a new family of Young diagrams.

Fig. 3. Four cases of coloring of columns.

Definition 7. Let c be a fixed positive integer. A *c-colored $2 \times n$ Young diagram* is a Young diagram of shape (n, n) such that each cell may be colored by one of the c colors.

When $c = 1$, we get an ordinary $2 \times n$ Young diagram. When $c = 2$, a cell may be colored by black or white. As illustrated in Fig. 3, for any 2-colored $2 \times n$ Young diagram, we use U, D, B and N to denote a column with a black cell on the top and a white cell at the bottom, a column with a white cell on the top and a black cell at the bottom, a column with two black cells and a column with two white cells, respectively. When $c = 3$, a cell may be colored by black, white or green. The *weight* of a Young diagram is defined to be the product of the weights of its cells, and the *weight* of a set of Young diagrams equals the sum of the weights of its Young diagrams.

Definition 8. We use CY_n to denote the subset of 2-colored $2 \times n$ Young diagrams such that the top row and bottom row have the same number of black cells.

Since the top row and bottom row have the same number of black cells, we see that the columns of U's and D's must be matched on any Young diagram in CY_n. It should be noted that Riordan [32, p. 81] proved the identity (14) by combining inverse relations and the generating function for Legendre polynomials. We are now ready to give an original proof of it.

Combinatorial proof of the identity (14):
For any Young diagram in CY_n, we assign the weight $x^{\frac{1}{2}}$ to each black cell and the weight 1 to each white cell. Consider a subset of CY_n consisting of all Young diagrams with exactly $2k$ black cells, i.e., the top row and bottom row both have exactly k black cells. Since $x = x^{\frac{1}{2}}x^{\frac{1}{2}}$ and there are $\binom{n}{k}$ ways to choose black cells from each row, the weight of this subset equals $\binom{n}{k}^2 x^k$. Thus the left-hand side of (14) equals the weight of CY_n. In particular, one has

$$\#\,\mathrm{CY}_n = \binom{2n}{n}. \tag{18}$$

As illustrated in Fig. 3, each column of Young diagrams in CY_n may be colored with the same color or different colors. Consider a subset of CY_n consisting of all Young diagrams having exactly k U's. Since the columns of U's and D's must be matched, there are $\binom{n}{2k}\binom{2k}{k}$ ways to locate all the U's and D's. The weight of the other columns is given by $(1+x)^{n-2k}$. Therefore, the weight of this subset equals

$$\binom{n}{2k}\binom{2k}{k}x^k(1+x)^{n-2k},$$

which is the summand of the right-hand side of (14). This completes the proof. □

Definition 9. Let TY_n be the subset of 3-colored $2\times n$ Young diagram such that

(*i*) The top row and bottom row have the same number of black cells;
(*ii*) There are five cases of coloring of columns, in addition to previous U, D, B and N, and a column may be two green cells and we use G to denote it.

Combinatorial proof of the identity (15):
For any Young diagram in CY_n, we first assign the weight x to each black cell and the weight $1+x$ to each white cell. Note that there are $\binom{n}{k}$ ways to choose black cells in each row. Then the weight of CY_n equals

$$\sum_{k=0}^{n}\left(\binom{n}{k}x^k(1+x)^{n-k}\right)^2=\sum_{k=0}^{n}\binom{n}{k}^2x^{2k}(1+x)^{2n-2k}.$$

Let $E(k)$ be any subset of CY_n with k U's, k D's and the positions of the U's and D's are fixed. Then $E(k)$ has the weight

$$(x(1+x))^{2k}(x^2+(1+x)^2)^{n-2k}.$$

For any path in TY_n, we assign the weight $x(1+x)$ to each of the U, D, B and N columns, and the weight 1 to each G column. Let $\widehat{E}(k)$ be any subset of $\mathrm{TY}_n$with k U's, k D's and the positions of the U's and D's are fixed. The weight of $\widehat{E}(k)$ equals

$$(x(1+x))^{2k}(1+x(1+x)+x(1+x))^{n-2k}.$$

Hence $E(k)$ and $\widehat{E}(k)$ have the same weight. For any Young diagram in TY_n with $n-k$ G columns, the remaining k columns form a new Young diagram in CY_k. It follows from (18) that the weight of the subset of Young diagrams in TY_n with $n-k$ G's equals

$$\binom{n}{n-k}1^{n-k}\binom{2k}{k}\left(x(1+x)\right)^k=\binom{n}{k}\binom{2k}{k}x^k(1+x)^k,$$

which is the summand of the right-hand side of (15). This completes the proof. □

Combinatorial proof of the identity (17):
For any Young diagram in CY_n, we reassign the weight x to each black cell and the weight 1 to each white cell. Consider the subset of CY_n consisting of all Young diagrams with exactly $2k$ black cells. The weight of this subset equals $\binom{n}{k}^2x^{2k}$. In the same way as the combinatorial proof of (14), the left-hand side of (17) equals the weight of CY_n.

In order to distinguish, we now use $H(k)$ to denote any subset of CY_n with k U's, k D's and the U's and D's are all located in given positions. The weight of $H(k)$ equals $x^{2k}(1+x^2)^{n-2k}$.

For any Young diagram in TY_n, we assign the weight $(-x)$ to each of the U, D, B and N columns, and the weight $(1+x)^2$ to each G column. Similarly, we use $\widehat{H}(k)$ to denote any subset of TY_n with k U's, k D's and the U's and D's are all located in given positions. Since $x^2=(-x)(-x)$ and $1+x^2=(1+x)^2-x-x$, the weight of $\widehat{H}(k)$ equals

$$x^{2k}\left((1+x)^2-x-x\right)^{n-2k}=x^{2k}(1+x^2)^{n-2k}.$$

Hence $\widehat{H}(k)$ and $H(k)$ have the same weight. It remains to show that the weight TY_n coincides with the right-hand side of (17). For any Young diagram in TY_n with $n-k$ G columns, the remaining k columns form a new Young diagram in CY_k. It follows from (18) that the weight of the subset of Young diagrams in TY_n with $n-k$ G's equals

$$\binom{n}{n-k}\left((1+x)^2\right)^{n-k}\binom{2k}{k}(-x)^k=\binom{n}{k}\binom{2k}{k}(-x)^k(1+x)^{2n-2k},$$

which is the summand of the right-hand side of (17). This completes the proof. □

3.3. *Dual formulas of* (7) *and* (8)

As pointed out by Petersen [29, Preface], the Narayana numbers are close cousins of the Eulerian numbers. We can now present the connections between differential operators and Narayana polynomials, which may be seen as dual formulas of (7) and (8).

Theorem 10. *For $n \geqslant 1$, we have*

$$\left(\frac{x^2}{1-x^2}D\right)^n\frac{1}{1-x^2}=\frac{(n+1)!x^{n+2}N(A_{n-1},x^2)}{(1-x^2)^{2n+1}}, \tag{19}$$

$$\left(\frac{x^2}{1-x^2}D\right)^n\frac{x}{1-x^2}=\frac{n!x^{n+1}N(B_n,x^2)}{(1-x^2)^{2n+1}}. \tag{20}$$

Therefore, we have

$$\left(\frac{x^2}{1-x^2}D\right)^n\frac{1}{1-x}=\frac{n!x^{n+1}\left(N(B_n,x^2)+(n+1)xN(A_{n-1},x^2)\right)}{(1-x^2)^{2n+1}}. \tag{21}$$

Proof. Note that

$$\frac{x^2}{1-x^2}D\frac{1}{1-x^2}=\frac{2x^3}{(1-x^2)^3},\quad \left(\frac{x^2}{1-x^2}D\right)^2\frac{1}{1-x^2}=\frac{3!x^4(1+x^2)}{(1-x^2)^5},$$

$$\frac{x^2}{1-x^2}D\frac{x}{1-x^2}=\frac{x^2(1+x^2)}{(1-x^2)^3},\quad \left(\frac{x^2}{1-x^2}D\right)^2\frac{x}{1-x^2}=\frac{2x^3(1+4x^2+x^4)}{(1-x^2)^5}.$$

Thus the identities hold for $n = 1, 2$.

Note that

$$xN(A_{n-1}, x) = x\sum_{i=0}^{n-1}\frac{1}{n}\binom{n}{i+1}\binom{n}{i}x^i = \sum_{k=1}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^k.$$

To prove the general formulas, we introduce

$$\widehat{N}(n,k) = (n+1)!\frac{1}{n}\binom{n}{k}\binom{n}{k-1},\ \widehat{M}(n,k) = n!\binom{n}{k}^2.$$

Following [25, Lemma 7], the numbers $\widehat{M}(n,k)$ and $\widehat{N}(n,k)$ satisfy the recursions:

$$\widehat{M}(n+1,k) = (n+1+2k)\widehat{M}(n,k) + (3n+3-2k)\widehat{M}(n,k-1),$$
$$\widehat{N}(n+1,k) = (n+2k)\widehat{N}(n,k) + (3n+4-2k)\widehat{N}(n,k-1).$$

We now perform the inductive step. Assume that (19) and (20) hold for $n = m$. Then when $n = m + 1$, we obtain

$$\left(\frac{x^2}{1-x^2}D\right)\frac{\sum_{k=1}^{m}\widehat{N}(m,k)x^{2k+m}}{(1-x^2)^{2m+1}}$$
$$= \frac{\sum_{k=1}^{m}(2k+m)\widehat{N}(m,k)x^{2k+m+1}(1-x^2) + 2(2m+1)\sum_{k=1}^{m}\widehat{N}(m,k)x^{2k+m+3}}{(1-x^2)^{2m+3}},$$
$$\left(\frac{x^2}{1-x^2}D\right)\frac{\sum_{k=0}^{m}\widehat{M}(m,k)x^{2k+m+1}}{(1-x^2)^{2m+1}}$$
$$= \frac{\sum_{k=0}^{m}(2k+m+1)\widehat{M}(m,k)x^{2k+m+2}(1-x^2) + 2(2m+1)\sum_{k=0}^{m}\widehat{M}(m,k)x^{2k+m+4}}{(1-x^2)^{2m+3}}.$$

Combining these two expressions with the recursions of $\widehat{M}(n,k)$ and $\widehat{N}(n,k)$, we get

$$\left(\frac{x^2}{1-x^2}D\right)^{m+1}\frac{1}{1-x^2} = \frac{x^{m+1}\sum_{k=1}^{m+1}\widehat{N}(m+1,k)x^{2k}}{(1-x^2)^{2m+3}},$$
$$\left(\frac{x^2}{1-x^2}D\right)^{m+1}\frac{x}{1-x^2} = \frac{x^{m+2}\sum_{k=0}^{m+1}\widehat{M}(m+1,k)x^{2k}}{(1-x^2)^{2m+3}},$$

as desired. Since

$$\left(\frac{x^2}{1-x^2}D\right)^n\frac{1}{1-x} = \left(\frac{x^2}{1-x^2}D\right)^n\frac{1}{1-x^2} + \left(\frac{x^2}{1-x^2}D\right)^n\frac{x}{1-x^2},$$

the proof of (21) follows immediately. This completes the proof. □

Example 11. When $n = 2$ and $n = 3$, the polynomials $N(B_n, x^2) + (n+1)xN(A_{n-1}, x^2)$ are respectively given as follows:

$$1 + 3x + 4x^2 + 3x^3 + x^4 = (1 + x)^2(1 + x + x^2),$$
$$1 + 4x + 9x^2 + 12x^3 + 9x^4 + 4x^5 + x^6 = (1 + x)^2(1 + 2x + 4x^2 + 2x^3 + x^4).$$

We can see that the above two polynomials are both symmetric. Furthermore, if we divide these polynomials by $(1 + x)^2$, the remainders are also symmetric.

Motivated by (8), we further study $N(B_n, x^2) + (n + 1)xN(A_{n-1}, x^2)$ in the next subsection.

3.4. *Hurwitz stability and alternating gamma-positivity*

Let RZ denote the set of real polynomials with only real zeros. Furthermore, denote by RZ(I) the set of such polynomials all of whose zeros are in the interval I. Following [18], we say that a polynomial $p(x) \in \mathbb{R}[x]$ is *standard* if its leading coefficient is positive. Suppose that $p(x), q(x) \in$ RZ, and the zeros of $p(x)$ are $\xi_1 \leqslant \cdots \leqslant \xi_n$, and that those of $q(x)$ are $\theta_1 \leqslant \cdots \leqslant \theta_m$. We say that $p(x)$ *interlaces* $q(x)$ if $\deg q(x) = 1 + \deg p(x)$ and the zeros of $p(x)$ and $q(x)$ satisfy

$$\theta_1 \leqslant \xi_1 \leqslant \theta_2 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_{n+1}.$$

We say that $p(x)$ *alternates left of* $q(x)$ if $\deg p(x) = \deg q(x)$ and their zeros satisfy

$$\xi_1 \leqslant \theta_1 \leqslant \xi_2 \leqslant \theta_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_n.$$

The reader is referred to [22] for the method of interlacing zeros.

Let $\mathbb{C}[x]$ denote the set of all polynomials in x with complex coefficients. A polynomial $p(x) \in \mathbb{C}[x]$ is *Hurwitz stable* if every zero of $p(x)$ is in the open left half plane, and $p(x)$ is *weakly Hurwitz stable* if every zero of $p(x)$ is in the closed left half of the complex plane, see [2] for details. The classical Hermite-Biehler theorem is given as follows.

Hermite-Biehler Theorem *([18, Theorem 3]). Let $f(x) = f^E(x^2)+xf^O(x^2)$ be a standard polynomial with real coefficients. Then $f(x)$ is weakly Hurwitz stable if and only if both $f^E(x)$ and $f^O(x)$ are standard, have only nonpositive zeros, and $f^O(x)$ interlaces or alternates left of $f^E(x)$. Moreover, $f(x)$ is Hurwitz stable if and only if $f(x)$ is weakly Hurwitz stable, $f(0) \neq 0$ and $\gcd(f^E(x), f^O(x)) = 1$.*

We can now present the following result.

Theorem 12. *For any $n \geqslant 1$, the polynomial $N(B_n, x^2) + (n + 1)xN(A_{n-1}, x^2)$ is alternatingly γ-positive, Hurwitz stable, and can be divided by $(1 + x)^2$.*

Proof. Immediate from Theorem 4, we then get

$$
\begin{aligned}
& N(B_n, x^2) + (n+1)xN(A_{n-1}, x^2) \\
&= (1+x)^{2n} + \sum_{k=1}^{n} \left(\binom{n}{k}\binom{2k}{k} - (n+1)C_k \binom{n-1}{k-1} \right) (-x)^k (1+x)^{2n-2k}.
\end{aligned} \tag{22}
$$

For $1 \leqslant k \leqslant n$, we can see that

$$
\frac{\binom{n}{k}\binom{2k}{k}}{(n+1)C_k\binom{n-1}{k-1}} = \frac{\frac{n}{k}\binom{n-1}{k-1}\binom{2k}{k}}{\frac{n+1}{k+1}\binom{2k}{k}\binom{n-1}{k-1}} = \frac{n(k+1)}{(n+1)k} \geqslant 1,
$$

and so each term in the expansion (22) contains $(1+x)^2$ as a factor. When $k = n$,

$$
\binom{n}{k}\binom{2k}{k} = (n+1)C_k\binom{n-1}{k-1},
$$

which implies that the $k = n$ term vanishes in (22). So $N(B_n, x^2) + (n+1)xN(A_{n-1}, x^2)$ is alternatingly γ-positive and can be divided by $(1+x)^2$.

Recall that

$$
N(A_n, x) = \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} x^k, \; N(B_n, x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k.
$$

We obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(xN(A_n, x)\right) &= \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} x^k \\
&= \sum_{k=0}^{n} \binom{n}{k}\binom{n+1}{k} x^k \\
&= \sum_{k=0}^{n} \binom{n}{k}^2 x^k + \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} x^k \\
&= N(B_n, x) + nxN(A_{n-1}, x), \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(N(B_n, x) + nxN(A_{n-1}, x)\right) &= \sum_{k=1}^{n} \binom{n}{k}\binom{n+1}{k} kx^{k-1} \\
&= n(n+1)N(A_{n-1}, x).
\end{aligned} \tag{23}
$$

According to [4, Corollary 7.2], we have $N(A_n, x) \in \mathrm{RZ}(-\infty, 0)$ and $N(B_n, x) \in \mathrm{RZ}(-\infty, 0)$. By *Rolle's theorem*, one can immediately get that $\frac{\mathrm{d}}{\mathrm{d}x}(xN(A_n, x)) \in \mathrm{RZ}(-\infty, 0)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xN(A_n,x)\right)=N(B_n,x)+nxN(A_{n-1},x),$$

so we have $N(B_n,x)+nxN(A_{n-1},x)\in \mathrm{RZ}(-\infty,0)$. Similarly, by *Rolle's theorem*, it follows from (23) that the polynomial $N(A_{n-1},x)$ interlaces $N(B_n,x)+nxN(A_{n-1},x)$.

Suppose that the zeros of $N(A_{n-1},x)$ are $r_{n-1}<r_{n-2}<\cdots<r_1$, and that those of $N(B_n,x)$ are $s_n<s_{n-1}<\cdots<s_1$. Since $N(A_{n-1},x)$ interlaces $N(B_n,x)+nxN(A_{n-1},x)$ and

$$\operatorname{sgn}\left(N(B_n,r_i)+nr_iN(A_{n-1},r_i)\right)=\operatorname{sgn}N(B_n,r_i),$$

we can see that the sign of $N(B_n,r_i)$ is $(-1)^i$ for each $i\in[n-1]$. Note that $N(B_n,x)$ is monic, $N(B_n,0)=1$ and $\operatorname{sgn}N(B_n,-\infty)=(-1)^n$. Hence $N(B_n,x)$ has precisely one zero in each of the n intervals $(-\infty,r_{n-1}),(r_{n-1},r_{n-2}),\ldots,(r_2,r_1),(r_1,0)$. Thus $N(A_{n-1},x)$ interlaces $N(B_n,x)$. Using the Hermite-Biehler theorem, we conclude that $N(B_n,x^2)+(n+1)xN(A_{n-1},x^2)$ is Hurwitz stable. This completes the proof. □

Inductively define the polynomials $L_n(x)$ and $\widehat{L}_n(x)$ by

$$\left(\frac{x^2}{1-x^2}D\right)^n\frac{1}{1-x}=\frac{n!x^{n+1}(1+x)^2L_n(x)}{(1-x^2)^{2n+1}}=\frac{n!x^{n+1}(1+x)\widehat{L}_n(x)}{(1-x^2)^{2n+1}}.$$

By induction, it is routine to deduce the following result.

Proposition 13. *For $n\geqslant 1$, we have*

$$nL_n(x)=(n+2x+(3n-4)x^2)L_{n-1}(x)+x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}L_{n-1}(x),$$
$$n\widehat{L}_n(x)=(n+x+(3n-3)x^2)\widehat{L}_{n-1}(x)+x(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\widehat{L}_{n-1}(x),$$

with the initial conditions $L_1(x)=1$ and $\widehat{L}_0(x)=1$.

By (21), we have $(1+x)^2L_n(x)=N(B_n,x^2)+(n+1)xN(A_{n-1},x^2)$ for $n\geqslant 1$. Therefore, by Theorem 12, we immediately get the following result.

Corollary 14. *Both $L_n(x)$ and $\widehat{L}_n(x)$ are alternatingly γ-positive and Hurwitz stable.*

From Proposition 13, we note that $nL_n(1)=(4n-2)L_{n-1}(1)$. Thus

$$2L_n(1)=\widehat{L}_n(1)=\binom{2n}{n}.$$

Below are the polynomials $L_n(x)$ for $n\leqslant 5$:

$$\begin{aligned}
&L_1(x)=1,\ L_2(x)=1+x+x^2,\ L_3(x)=1+2x+4x^2+2x^3+x^4,\\
&L_4(x)=1+3x+9x^2+9x^3+9x^4+3x^5+x^6,\\
&L_5(x)=1+4x+16x^2+24x^3+36x^4+24x^5+16x^6+4x^7+x^8.
\end{aligned}$$

It should be noted that the sequences $\{L(n,k)\}_{k=0}^{2n-2}$ and $\{\widehat{L}(n,k)\}_{k=0}^{2n-1}$ appear as A088855 in [36], which count symmetric Dyck paths by their number of peaks. These sequences have been discussed recently by Cho, Huh and Sohn [10, Lemma 3.8]. Explicitly, we have

$$L(n,k)=\binom{n-1}{\lceil\frac{k}{2}\rceil}\binom{n-1}{\lfloor\frac{k}{2}\rfloor},\ \widehat{L}(n,k)=\binom{n}{\lceil\frac{k}{2}\rceil}\binom{n-1}{\lfloor\frac{k}{2}\rfloor}$$

which can be directly verified by using Proposition 13.

4. Identities involving Eulerian polynomials

In [3], Brändén introduced the following modified Foata-Strehl action (MFS -action for short), which can be used to show the γ-positivity of various enumerative polynomials.

MFS-action *([3]). Given $\pi\in\mathfrak{S}_n$ and $x=\pi(i)$.*

- (i) *If x is a double descent, then let $\varphi_x(\pi)$ be obtained by deleting x and then inserting x between $\pi(j)$ and $\pi(j+1)$, where j is the smallest index satisfying $j>i$ and $\pi(j)<x<\pi(j+1)$;*
- (ii) *If x is a double ascent, then let $\varphi_x(\pi)$ be obtained by deleting x and then inserting x between $\pi(j)$ and $\pi(j+1)$, where j is the largest index satisfying $j<i$ and $\pi(j)>x>\pi(j+1)$;*
- (iii) *If x is a peak or a valley, then let $\varphi_x(\pi)=\pi$.*

For each $x\in[n]$, the MFS-action is defined by

$$\varphi'_x(\pi)=\begin{cases}\varphi_x(\pi), & \text{if } x \text{ is a double ascent or double descent;}\\ \pi, & \text{if } x \text{ is a valley or a peak.}\end{cases}$$

The reader is referred to [3, Fig. 1] for an visualized instance of the modified Foata–Strehl action. It is clear that φ'_x's are involutions and that they commute. For any $S\subseteq[n]$, the function $\varphi'_S:\mathfrak{S}_n\to\mathfrak{S}_n$ is defined by

$$\varphi'_S(\pi)=\prod_{x\in S}\varphi'_x(\pi).$$

Therefore, the group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions φ'_S, where $S\subseteq[n]$.

Let $\mathrm{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\}$ be the orbit of π under the MFS-action. Brändén noted that the following result follows from the work in [15], and he proved it by using the MFS-action.

Proposition 15 *([3, Theorem 3.1]). For any $\pi \in \mathfrak{S}_n$, one has*

$$\sum_{\sigma\in\mathrm{Orb}(\pi)} x^{\mathrm{des}\,(\sigma)} = x^{\mathrm{des}\,(\widehat{\pi})}(1+x)^{n-1-2\mathrm{des}\,(\widehat{\pi})} = x^{\mathrm{pk}\,(\pi)}(1+x)^{n-1-2\mathrm{pk}\,(\pi)}, \tag{24}$$

where $\widehat{\pi}$ to denote the unique element in $\mathrm{Orb}(\pi)$ with no double descents.

An immediate consequence of (24) is the following result.

Proposition 16. *For any $\pi \in \mathfrak{S}_n$, one has*

$$\sum_{\sigma\in\mathrm{Orb}(\pi)} x^{2\mathrm{des}\,(\sigma)} = \sum_{i=0}^{n-1-2\mathrm{pk}\,(\pi)} \binom{n-1-2\mathrm{pk}\,(\pi)}{i} 2^i(-x)^{2\mathrm{pk}\,(\pi)+i}(1+x)^{2n-2-2(2\mathrm{pk}\,(\pi)+i)}.$$

The *peak polynomials* and *left peak polynomials* are respectively defined by

$$P_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{pk}\,(\pi)} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} P(n,k)x^k,\ \widehat{P}_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{lpk}\,(\pi)} = \sum_{k=0}^{\lfloor n/2\rfloor} \widehat{P}(n,k)x^k.$$

They satisfy the following recurrence relations

$$P_{n+1}(x) = (nx - x + 2)P_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x), \tag{25}$$

$$\widehat{P}_{n+1}(x) = (nx+1)\widehat{P}_n(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}\widehat{P}_n(x), \tag{26}$$

with the initial values $P_1(x) = \widehat{P}_1(x) = 1$, $P_2(x) = 2$ and $\widehat{P}_2(x) = 1 + x$. These polynomials arise often in algebra, combinatorics and other branches of mathematics, see [20,23,37,38].

By Theorem 2, we can now conclude the following result.

Theorem 17.

(*i*) *For $n \geqslant 1$, both $A_n(x^2)$ and $B_n(x^2)$ are alternatingly γ-positive. More precisely, there exist nonnegative integers $a(n,k)$ and $b(n,k)$ such that*

$$A_n(x^2) = \sum_{k=0}^{n-1} a(n,k)(-x)^k(1+x)^{2n-2-2k},\ B_n(x^2) = \sum_{k=0}^{n} b(n,k)(-x)^k(1+x)^{2n-2k}.$$

(*ii*) *For* $n \geqslant 1$, *we have*

$$(1+x)^{2n-2} A_n\left(\frac{x^2}{(1+x)^2}\right) = \sum_{k=0}^{n-1} a(n,k) x^k (1+x)^k,$$

$$(1+x)^{2n} B_n\left(\frac{x^2}{(1+x)^2}\right) = \sum_{k=0}^{n} b(n,k) x^k (1+x)^k.$$

(*iii*) *Setting* $a_n(x) = \sum_{k=0}^{n-1} a(n,k)x^k$ *and* $b_n(x) = \sum_{k=0}^{n} b(n,k)x^k$, *then we get*

$$a_n(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 4^k P(n,k) x^{2k} (1+2x)^{n-1-2k}$$
$$= \left(\frac{1+2x}{2}\right)^{n-1} P_n\left(\frac{4x^2}{(1+2x)^2}\right),$$

$$b_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^k \widehat{P}(n,k) x^{2k} (1+2x)^{n-2k} = (1+2x)^n \widehat{P}_n\left(\frac{4x^2}{(1+2x)^2}\right).$$

(*iv*) *There exist nonnegative integers* $\alpha(n,i)$ *and* $\beta(n,i)$ *such that*

$$a_n(x) = \sum_{i=0}^{n-1} \alpha(n,i) x^i (1+x)^{n-1-i}, \; b_n(x) = \sum_{i=0}^{n} \beta(n,i) x^i (1+x)^{n-i},$$

$$\frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 4^k P(n,k) x^{2k} = \sum_{i=0}^{n-1} \alpha(n,i)(-x)^i (1+x)^{n-1-i}, \tag{27}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} 4^k \widehat{P}(n,k) x^{2k} = \sum_{i=0}^{n} \beta(n,i)(-x)^i (1+x)^{n-i}. \tag{28}$$

From Proposition 3, we see that

$$a(n,k) = \frac{1}{2^{n-1-k}} \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-1-2i}{k-2i} P(n,i), \; b(n,k) = 2^k \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-2i}{k-2i} \widehat{P}(n,i).$$

For $n \geqslant 1$, we now define

$$\alpha_n(x) = \sum_{i=0}^{n-1} \alpha(n,i) x^i, \; \beta_n(x) = \sum_{i=0}^{n} \beta(n,i) x^i.$$

The reader is referred to Table 1 for the initial values of $a_n(x), b_n(x), \alpha_n(x)$ and $\beta_n(x)$.

Setting $y = \frac{-x}{1+x}$ in (27) and (28), we get the following corollary.

Corollary 18. *For $n \geqslant 1$, one has*

$$\alpha_n(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 4^k P(n,k) x^{2k}(1+x)^{n-1-2k} = \left(\frac{1+x}{2}\right)^{n-1} P_n\left(\frac{4x^2}{(1+x)^2}\right), \tag{29}$$

$$\beta_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^k \widehat{P}(n,k) x^{2k}(1+x)^{n-2k} = (1+x)^n \widehat{P}_n\left(\frac{4x^2}{(1+x)^2}\right).$$

Corollary 19. *The polynomials $a_n(x), b_n(x), \alpha_n(x)$ and $\beta_n(x)$ satisfy the recursions*

$$a_{n+1}(x) = (1+3x-nx)a_n(x) + \frac{1}{2}x(1+4x)\frac{\mathrm{d}}{\mathrm{d}x}a_n(x), \tag{30}$$

$$b_{n+1}(x) = (1+2x-2nx)b_n(x) + x(1+4x)\frac{\mathrm{d}}{\mathrm{d}x}b_n(x), \tag{31}$$

$$\alpha_{n+1}(x) = \left(1+x+\frac{1}{2}(n-1)x(3x-1)\right)\alpha_n(x) + \frac{1}{2}x(1-x)(1+3x)\frac{\mathrm{d}}{\mathrm{d}x}\alpha_n(x), \tag{32}$$

$$\beta_{n+1}(x) = (1+x-nx+3nx^2)\beta_n(x) + x(1-x)(1+3x)\frac{\mathrm{d}}{\mathrm{d}x}\beta_n(x), \tag{33}$$

with the initial conditions $a_1(x) = \alpha_1(x) = b_0(x) = \beta_0(x) = 1$.

Proof. Differentiation of

$$P_n\left(\left(\frac{2x}{1+2x}\right)^2\right) = \left(\frac{2}{1+2x}\right)^{n-1} a_n(x)$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n\left(\left(\frac{2x}{1+2x}\right)^2\right) = \frac{2^{n-4}(1+2x)\frac{\mathrm{d}}{\mathrm{d}x}a_n(x) - 2^{n-3}(n-1)a_n(x)}{x(1+2x)^{n-3}}.$$

Substituting these two expressions into (25), we get (30). Differentiation of

$$\widehat{P}_n\left(\left(\frac{2x}{1+2x}\right)^2\right) = \frac{b_n(x)}{(1+2x)^n}$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\widehat{P}_n\left(\left(\frac{2x}{1+2x}\right)^2\right) = \frac{(1+2x)\frac{\mathrm{d}}{\mathrm{d}x}b_n(x) - 2nb_n(x)}{8x(1+2x)^{n-2}}.$$

Substituting the above two expressions into (26) and simplifying, we obtain (31).

Differentiation of

$$P_n\left(\left(\frac{2x}{1+x}\right)^2\right)=\left(\frac{2}{1+x}\right)^{n-1}\alpha_n(x)$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n\left(\left(\frac{2x}{1+x}\right)^2\right)=\frac{2^{n-4}(1+x)\frac{\mathrm{d}}{\mathrm{d}x}\alpha_n(x)-2^{n-4}(n-1)\alpha_n(x)}{x(1+x)^{n-3}}.$$

Substituting these two expressions into (25), we get (32). Differentiation of

$$\widehat{P}_n\left(\left(\frac{2x}{1+x}\right)^2\right)=\frac{\beta_n(x)}{(1+x)^n}$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\widehat{P}_n\left(\left(\frac{2x}{1+x}\right)^2\right)=\frac{(1+x)\frac{\mathrm{d}}{\mathrm{d}x}\beta_n(x)-n\beta_n(x)}{8x(1+x)^{n-2}}.$$

Substituting the above two expressions into (26) and simplifying, we arrive at (33). □

From (32), we see that $\alpha_n(1)=n!$. Set $\widehat{\alpha}_0(x)=1$ and $\widehat{\alpha}_n(x)=x^{n-1}\alpha_n\left(\frac{1}{x}\right)$ for $n\geqslant 1$. Let $\gamma_{n,k}=\#\{\pi\in\mathfrak{S}_n:\ \mathrm{pk}\,(\pi)=k,\ \mathrm{ddes}\,(\pi)=0\}$. Combining (3), (4) and (29), we get

$$\begin{aligned}\widehat{\alpha}_n(x)&=\left(\frac{1+x}{2}\right)^{n-1}P_n\left(\left(\frac{2}{1+x}\right)^2\right)\\&=\frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}4^kP(n,k)(1+x)^{n-1-2k}\\&=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\gamma_{n,k}(1+x)^{n-1-2k}.\end{aligned}$$

By using the MFS-action defined by (4), one can immediately get that $\widehat{\alpha}_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{dasc}\,(\pi)}$, since each double ascent of π can be transformed to a double descent. It is well known [36, A008303] that the exponential generating function of peak polynomials is given as follows:

$$P(x;z):=\sum_{n=1}^{\infty}P_n(x)\frac{z^n}{n!}=\frac{\sinh(z\sqrt{1-x})}{\sqrt{1-x}\cosh(z\sqrt{1-x})-\sinh(z\sqrt{1-x})}.\tag{34}$$

Note that

Table 1
The initial values of $a_n(x), b_n(x), \alpha_n(x)$ and $\beta_n(x)$.

$a_1(x) = 1$	$a_2(x) = 1 + 2x$	$a_3(x) = 1 + 4x + 6x^2$
$b_1(x) = 1 + 2x$	$b_2(x) = 1 + 4x + 8x^2$	$b_3(x) = 1 + 6x + 32x^2 + 48x^3$
$\alpha_1(x) = 1$	$\alpha_2(x) = 1 + x$	$\alpha_3(x) = 1 + 2x + 3x^2$
$\beta_1(x) = 1 + x$	$\beta_2(x) = 1 + 2x + 5x^2$	$\beta_3(x) = 1 + 3x + 23x^2 + 21x^3$

$$\widehat{\alpha}(x;z) := \sum_{n=0}^{\infty} \widehat{\alpha}_n(x)\frac{z^n}{n!} = 1 + \frac{2}{1+x}P\left(\left(\frac{2}{1+x}\right)^2; \frac{(1+x)z}{2}\right). \tag{35}$$

Set $u = \sqrt{(x+3)(x-1)}$. Combining (34) and (35), it is routine to check that

$$\widehat{\alpha}(x;z) = \frac{u\cosh\left(\frac{1}{2}uz\right) + (1-x)\sinh\left(\frac{1}{2}uz\right)}{u\cosh\left(\frac{1}{2}uz\right) - (1+x)\sinh\left(\frac{1}{2}uz\right)},$$

which was also recently studied by Zhuang [38, Theorem 13]. In conclusion, we can now restate Corollary 18 in a succinct form as follows.

Proposition 20. *For $n \geqslant 1$, we have*

$$\alpha_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{n-1-\mathrm{dasc}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{pk}\,(\pi)+\mathrm{des}\,(\pi)},$$
$$\beta_n(x) = \sum_{\pi\in\mathfrak{S}_n} (2x)^{2\mathrm{lpk}\,(\pi)}(1+x)^{n-2\mathrm{lpk}\,(\pi)}.$$

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